Int. J. Solids Structures, 1975, Vol. 11, pp. 1219–1221. Pergamon Press. Printed in Great Britain

TORSIONAL STIFFNESS OF A LAYER BONDED TO AN ELASTIC HALF SPACE

H. H. JABALI
Senior Structural Engineer, Sargent & Lundy, Chicago, IL 60603, U.S.A.

(Received 17 March 1975)

Abstract—The static solution to the problem of a layer bonded to an elastic half-space, where the layer is driven by the torsional rotation of a bonded rigid circular disk, is considered here. An iterative solution, perturbing on that given for the elastic half-space, is obtained as a convergent power series, provided the ratio of the stratum depth to the radius of the disk is large. An equation for the applied static torque at the surface of the rigid disk is also calculated and compared, under limiting cases, with known results.

1. INTRODUCTION

The axially symmetric problem of the torsional oscillations of a rigid circular disk of radius a attached to an elastic layer bonded to an elastic half-space was previously considered in a separate paper by the author and colleagues[1]. Integral transforms techniques were used to formulate the problem in a manner similar to that used by Gladwell[2] for the case of forced torsional vibrations of an elastic stratum. In both instances, the problem was reduced to a Fredholm integral equation of the second kind, which Gladwell solved by iteration. His equations for the dynamic as well as static torque were given in the form of a power series.

The purpose of the present analysis is to supplement the results given in Ref. [1] by developing an iterative solution for the static case.

2. BASIC EQUATIONS

The static case of the problem of a layer bonded to an elastic half-space, where the layer is driven by torsional oscillations of a bonded rigid circular disk of radius a (see, e.g. [1]), is considered here. The thickness of the layer is h and the geometry and coordinate system are shown in Fig. 1. The shear moduli of the layer and foundation are denoted by μ_1 and μ_2 respectively. In Ref. [1], the problem was shown to reduce to a Fredholm integral equation of the second kind given by:

$$\theta(x)+\frac{1}{\pi}\int_0^1 M^*(x,\xi)\theta(\xi)\,d\xi = 2\phi x \quad 0\leq x\leq 1 \tag{1}$$

where ϕ is the angle of twist, $\theta(x)$ is an unknown auxiliary function, and the kernel, $M^*(x,\xi)$, was greatly simplified for the static case to:

$$M^*(x,\xi)=2\int_0^\infty [1+\alpha e^{2dt}]^{-1}\{\cos[(x+\xi)t]-\cos[(x-\xi)t]\}\,dt \tag{2}$$

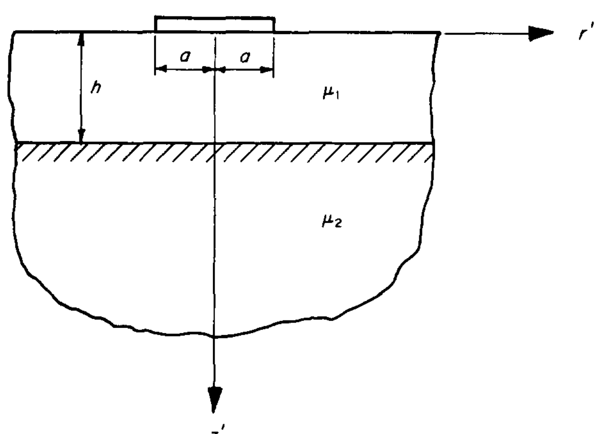

Fig. 1. Geometry and coordinate system.

where:

$$d = h/a, \quad \delta = \mu_2/\mu_1, \quad \alpha = \frac{\delta+1}{\delta-1}. \tag{3}$$

The important physical quantity in this problem is the applied torque at the surface of the disk, which in terms of the unknown auxiliary function, can be calculated as:

$$T = 8\mu_1 a^3 \int_0^1 t\theta(t)\,dt. \tag{4}$$

3. SOLUTION

An iterative solution of the integral equation for $\theta(x)$ giving a solution perturbing on that for a half-space will now be developed. The unknown function $\theta(x)$ appearing in eqn (1), the kernel M^* and the free term $H[H(x) = 2\phi x]$ are written, for large values of d, in the following form:

$$H(x) = \sum_{n=0}^{\infty} d^{-n} H_n(x) \tag{5}$$

$$\theta(x) = \phi \sum_{n=0}^{\infty} d^{-n}\theta_n(x) \tag{6}$$

$$M^*(x,\xi) = \sum_{n=1}^{\infty} d^{-n} M_n^*(x,\xi) \tag{7}$$

where it is assumed that $M_0^*(0,\xi) = M_0^*(x,0) = 0$. The integral in eqn (1) can be expanded in powers of d since

$$(1+\alpha\,e^{2td})^{-1} = \sum_{n=1}^{\infty} (-1)^{n-1}\alpha^{-n} e^{-2ntd} \quad (td \neq 0) \tag{8}$$

and

$$\int_0^{\infty} e^{-2ntd}\{\cos(x+\xi)t - \cos(x-\xi)t\}\,dt = \sum_{m=1}^{\infty} (-1)^m \frac{(x+\xi)^{2m} - (x-\xi)^{2m}}{(2nd)^{2m+1}}. \tag{9}$$

Thus, eqn (2) for $M^*(x,\xi)$ can be expressed in the form of eqn (7) as

$$M^*(x,\xi) = \sum_{m=1}^{\infty} (-1)^m \frac{(x+\xi)^{2m} - (x-\xi)^{2m}}{2^{2m}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\alpha^{-n}}{n^{2m+1}} d^{-(2m+1)}. \tag{10}$$

The coefficients M_n^* appearing in eqn (7) are now easily obtained by direct comparison of eqns (7) and (10) and are given by the relation

$$M_{2m+1}^* = \sum_{m=1}^{\infty} (-1)^m \frac{(x+\xi)^{2m} - (x-\xi)^{2m}}{2^{2m}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\alpha^{-n}}{n^{2m+1}} \tag{11}$$

$$(M_1 = M_2 = M_{2m} = 0).$$

By substituting eqns (5), (6) and (7) into eqn (1) and using an elementary relationship on the Cauchy product of two series[3], we obtain

$$\sum_{n=0}^{\infty} d^{-n}\theta_n(x) + \frac{1}{\pi}\int_0^1 \sum_{j=1}^{\infty} d^{-j} \sum_{n=0}^{j-1} M_{j-n}^*(x,\xi)\theta_n(\xi)\,d\xi = \sum_{n=0}^{\infty} d^{-n} H_n(x) \tag{12}$$

where M_{j-n}^* are given by (11).

By equating like powers of d on both sides of (12), the iterative solution for $\theta(x)$ is obtained as

$$\theta(x)=\phi\left[2x+\frac{2x}{2\pi d^3}R_3-\frac{(5x^3+3x)}{15\pi d^5}R_5+\frac{2x}{9d^6}\left(\frac{R_3}{\pi}\right)^2+0(d^{-7})+\ldots+\right] \tag{13}$$

where

$$R_{2m+1}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\alpha^{-n}}{n^{2m+1}}. \tag{14}$$

Finally, substitution of eqn (13) into eqn (4) yields a powers series solution in terms of d for the total applied static torque at the surface of the disc given by

$$T=\frac{16\mu_1 a^3\phi}{3}\left[1+\frac{1}{3\pi d^3}R_3-\frac{1}{5\pi d^5}R_5+\frac{1}{9\pi^2 d^6}(R_3)^2+0(d^{-7})+\ldots+\right]. \tag{15}$$

Limiting cases

The solution for a *rigid foundation* can be obtained from eqn (15) by taking the limiting values of R_{2m+1} as δ approaches infinity. The value of α therefore becomes unity and eqn (14) becomes

$$R_{2m+1}=(1-2^{-2m})\zeta(2m+1) \tag{16}$$

where $\zeta(2m+1)$ is the Riemann Zeta function[4]. Using the values of R_3 and R_5 from eqn (16) the torque becomes:

$$T=\frac{16\mu_1 a^3\phi}{3}\left[1+\frac{\zeta(3)}{4\pi d^3}-\frac{3\zeta(5)}{16\pi d^5}+\frac{\zeta^2(3)}{16\pi^2 d^6}+0(d^{-7})+\ldots+\right]. \tag{17}$$

The *free boundary* solution corresponds to the case when the foundation has zero stiffness with respect to the layer. The solution can be obtained by taking the limiting value of R_{2m+1} in eqn (14) as δ approaches zero, i.e. for $\alpha=-1$. Thus the torque becomes

$$T=\frac{16\mu_1 a^3\phi}{3}\left[1-\frac{\zeta(3)}{3\pi d^3}+\frac{\zeta(5)}{3\pi d^5}+\frac{\zeta^2(3)}{9\pi^2 d^6}+0(d^{-7})+\ldots+\right]. \tag{18}$$

Equations (17) and (18) are in agreement with the results given by Gladwell[2].

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